A Compendium of Classical Quaternion and Octonion Formulae
— a working paper

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Summary

We collect together in this *incomplete* working paper many formulae applicable to classical quaternions and octonions over the real and complex fields. Each formula is referenced to an original or definitive source wherever possible, and some minimal explanation and interpretation is given. With a few exceptions, proofs are omitted because they are available in the referenced sources. Where this is not the case, a derivation or proof is given in the Appendix. Notes are included in the text like this indicating additional material which is known but not yet properly included and cited.

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Preface

This collection of formulae relate to what some call 'classical' quaternions and octonions, meaning the two algebras that were known to Hamilton in the 1840s, rather than more modern generalisations over arbitrary fields. It is, however, assumed that the coefficients of the quaternions and octonions may be complex as well as real, unless stated otherwise.

The intent of the collection is to be useful as a reference for those who wish to manipulate quaternionic and octonionic algebraic expressions. The problem with this is that although most familiar rules of algebra apply, some don't. Neither quaternion nor octonion multiplication is commutative, meaning that pq is not necessarily equal to qp. So, how can you change the ordering? More precisely, what other change do you have to make in order not to alter the result of the product of two terms if you reverse their order? The answer will be found in the generalized conjugate rule given in § 2.9.

Octonions are a much more difficult case, since they also are not associative, meaning that a product of three or more terms may not yield the same result for every possible order of evaluation. Thus $p(qr)$ is not necessarily equal to $(pq)r$, the parentheses indicating which pair of octonions must be multiplied together first. This is a much greater obstacle than non-commutativity, which is concerned with left/right ordering of each multiplication. *Bi-associativity* (§ 4.1) and the famous Moufang formulae (§ 4.2) are important tools for dealing with this issue, plus of course the various formulae that apply to quaternions and which generalise to the octonion case (including the generalized conjugate rule mentioned in the previous paragraph).

Many of the formulae in this document look straightforward, if not trivial, but they are or were not easy to discover, as was noted by J. L. Synge [49, p 34] in 1972:

> It is typical of quaternion formulae that, though they be difficult to find, once found they are immediately verifiable.

Synge's statement is even more pertinent to octonion formulae, although some of them take the same form as the corresponding quaternion formula.

The formulae in this document have been verified using a Matlab®toolbox [40] (this is primarily a numerical toolbox, although version 3 has a symbolic capability for both quaternions and octonions). Unfortunately the author does not have access to Mathematica, which looks to be the best computer algebra system. There is a very powerful Maple package [1] which is much more general in its scope, but its use seems to require considerable mathematical expertise that I don't have.

I thank Dr Todd Ell for making available to me his private quaternion 'cheat sheet' (back in the 1990s), and Dr Eckhard Hitzer for suggesting in 2010 that it would be very helpful to have as many formulae as possible collected together into a freely-available document. Realisation of the suggestion took over a decade simply because there were many other things that took priority. For 10 years this document existed as a rough draft, until I realised that the best way to complete it would be to issue a partially completed version as a working paper, and then incrementally update it as time and inclination permitted. My source document has additional notes apart from the ones that appear here in blue. These notes are not yet ready for public view, but they mean that there is more to the document in future than what you see here.

More than two years elapsed between revisions 2 and 3. Material was added and re-ordered, but not incrementally as I had hoped. Nothing much happened other than insertion of new private notes here and there, until January 2025 because of other projects that required my attention.

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1 Introduction

In this work, *quaternion* means a hypercomplex number (a generalization of the complex numbers to four dimensions), with a real or complex coefficient for each dimension. The *octonions* similarly are generalizations to eight dimensions.

It is not the purpose of this document to present an introduction to quaternions or octonions since introductions can be found elsewhere, and there is no need to duplicate them here. See, for example, Ward's book [54] which covers both, or for a more detailed introduction to octonions, consider the article by Baez [2].

This document is, however, intended to be somewhat more than a quaternion and octonion *cheat sheet*. There is an intentional tutorial aspect to the presentation, especially in the material on octonions, and this is particularly true of the Appendices.

1.1 Notation, terminology, and elementary formulae

The terminology of quaternions and octonions predates the modern concept of *vectors*. The *scalar part* of a quaternion or octonion is the coefficient that specifies its magnitude along the scalar axis of 4-space or 8-space (the equivalent in the complex numbers is the real part, of course). The *vector part* of a quaternion/octonion is the rest: the composite of the coefficients and basis elements, or more precisely, what is left after the scalar part is removed. The equivalent in the complex numbers is the imaginary part, but of course this is one-dimensional in the complex numbers.

Text in blue, as here, indicates something to be expanded, inserted, verified, *etc.* In a later version of the document the blue text should be replaced with more detailed, amended, or verified material.

Readers should be aware that there are many different notations used in the literature, even in comparatively recent papers. Totally different notations were used by early authors. Hamilton, for example, expressed the quaternion conjugate using a prefix K, so that in his notation, what is expressed in this document as $\overline{x}$ was written Kx. Coxeter retained some elements of this notation as late as 1946, writing Nx for the norm, where we would today write $\|x\|$.

As far as possible uniform notation is used throughout this document. The notation used may differ from the original source documents which are cited. The list following sets out the conventions used in this document, and also introduces some very basic quaternion formulae and definitions.

Fonts Full quaternions and octonions (*i.e.*, those with non-zero scalar part) are written as ordinary variables in italic, thus q; pure quaternions or octonions (with zero scalar part) as bold Roman or Greek variables, in colour, thus $\boldsymbol{f}$ or $\boldsymbol{\mu}$; and matrices in sans-serif upright font thus: P. We reserve **bold** Greek letters for the special case of roots of -1, that is pure quaternions or octonions which square to -1, with the exception of the basis elements $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ *etc.*

Cartesian form A quaternion in Cartesian form is $q = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the usual quaternion basis elements. We refer to the real or complex values w, x, y, z as the *coefficients* of the quaternion. An octonion in Cartesian form is similarly written $p = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} + a\boldsymbol{l} + b\boldsymbol{m} + c\boldsymbol{n} + d\boldsymbol{o}$, where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{n}, \boldsymbol{o}$ are the 7 octonion basis elements. We discuss the multiplication rules for the quaternion and octonion basis elements in § 2.1.

Complex root of -1 We use I for the complex root of -1 in order to distinguish it from the quaternion or octonion $\boldsymbol{i}$. Note that I commutes with all of the quaternion and octonion basis elements $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ *etc.*, and therefore with all quaternions and octonions with real or complex coefficients.

Scalar and vector parts The notation $\mathrm{S}(q) = w$ denotes the scalar part of a quaternion or octonion, and $\mathbf{V}(q) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ denotes the vector part of a quaternion, $\mathbf{V}(p) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} + a\boldsymbol{l} + b\boldsymbol{m} + c\boldsymbol{n} + d\boldsymbol{o}$ denotes the vector part of an octonion.

Pure and full A quaternion/octonion with zero scalar part is called *pure* (conversely *full* for the case where the scalar part is non-zero). This terminology dates right back to Hamilton, and is still used. Beware that some authors use *real* or *imaginary* for the scalar and vector parts, and thus replace *pure* with *imaginary*. We avoid doing this because it conflicts with the concept of real and imaginary in quaternions and octonions with complex coefficients.

Conjugation Quaternion conjugation (negation of the vector part) is denoted by an overbar: $\overline{q} = w - x\boldsymbol{i} - y\boldsymbol{j} - z\boldsymbol{k} = \mathrm{S}(q) - \mathbf{V}(q)$, and similarly for octonions. Complex conjugation (where the coefficients are complex) is indicated by a superscript star: $q^\star = w^\star + x^\star\boldsymbol{i} + y^\star\boldsymbol{j} + z^\star\boldsymbol{k}$. It is possible to apply both conjugates, denoted obviously by $\overline{q^\star}$. Unlike the complex conjugate, which cannot be expressed in terms of other operations, the quaternion/octonion conjugate can be written in terms of involutions (see § 2.11) using only arithmetic operations, and a constant orthonormal basis (see (2.50)).

Inner product The inner, or *scalar*, product of two quaternions is denoted $\langle p, q\rangle$ [54, § 2.5] and can be computed as the sum of the products of corresponding coefficients of p and q. This is a geometric scalar product in 4-space equivalent to treating the four coefficients of the quaternion as components of a 4-space vector. Similarly for octonions, the eight coefficients are treated as components of an 8-space vector. For pure quaternions or octonions, obviously, the inner product is in 3 or 7 dimensions respectively.

The scalar product may also be denoted by the centred dot (usually for pure quaternions) thus: $\boldsymbol{f} \cdot \boldsymbol{g}$. It is therefore also known as the 'dot' product.

See also § 2.7.

Norm and modulus Usage of these terms varies between authors. In this document, the *norm* of a quaternion is the sum of the squares of the coefficients: $\|q\| = w^2 + x^2 + y^2 + z^2$, and is also given by $\langle q, q\rangle$. The *modulus* is the square root of the norm: $|q| = \sqrt{\|q\|}$. When the coefficients are complex the norm may be complex, and is strictly referred to as a *semi-norm* [39, § 3]. These concepts extend to octonions without difficulty and the notation used is the same.

Inverse The complex numbers, quaternions and octonions are the only normed division algebras over $\mathbb{R}$ (this is proven by the Theorem of Frobenius [54, § 2.4]). In a normed algebra, the norm of a product is the product of the norms, thus: $\|pq\| = \|p\|\,\|q\|$. Every non-zero quaternion or octonion with real coefficients has a multiplicative inverse given by $q^{-1} = \overline{q}/\,\|q\|$. This is not necessarily true in the case of quaternions or octonions with complex coefficients — it is possible for a non-zero quaternion or octonion in this case to have a vanishing norm. Such a quaternion or octonion is called a *divisor of zero*. For a detailed discussion on this issue see [38].

The multiplicative inverse of a quaternion or octonion is a two-sided inverse: $q^{-1}q = qq^{-1} = 1$.

Sets We denote the sets of real numbers by $\mathbb{R}$, complex numbers by $\mathbb{C}$, quaternions by $\mathbb{H}$ and octonions by $\mathbb{O}$. Where we need to distinguish complexified quaternions and octonions from those with real coefficients, we will use English.

2 Quaternion formulae

Many formulae applicable to quaternions are also valid for octonions without modification. This is particularly so for some formulae involving only a pair of quaternions or octonions. A simple explanation for this is the geometrical isomorphism between two quaternions in a plane, and two octonions in a plane with the same angle between the two quaternions and the two octonions[1]. This geometry is isomorphic to a pair of complex numbers with, again, the same angle between them. Remember of course, that in the complex case, the two complex numbers commute, whereas the quaternions and octonions do not.

2.1 Product of two quaternions/octonions

To define the product of two quaternions in terms of Cartesian coefficients, we require the following famous rule discovered by Hamilton in 1843 [19]:

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1 \tag{2.1}$$

which is sufficient to define the full multiplication table[2] for quaternions as shown in the upper left quadrant of Table 2.1.

The product of two quaternions q_1 and q_2 is then:

$$q_1 q_2 = \begin{array}{l} w_1 w_2 - (x_1 x_2 + y_1 y_2 + z_1 z_2) \\ +(w_1 x_2 + x_1 w_2 + y_1 z_2 - z_1 y_2)\,\boldsymbol{i} \\ +(w_1 y_2 - x_1 z_2 + y_1 w_2 + z_1 x_2)\,\boldsymbol{j} \\ +(w_1 z_2 + x_1 y_2 - y_1 x_2 + z_1 w_2)\,\boldsymbol{k} \end{array} \tag{2.2}$$

where the cyan terms are the dot product of the vector parts, and the magenta terms *and signs* are the cross product of the vector parts. All the other terms on the RHS contain one or both scalar parts.

To define the multiplication table for octonions it is necessary to introduce a fourth root of -1, which we here denote by $\boldsymbol{l}$. We must then define the product of $\boldsymbol{l}$ with each of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. These products then define three more roots of -1 which we denote by $\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{o}$. Many permutations are possible, in addition to those already possible for the quaternions. We choose here to define $\boldsymbol{il} = \boldsymbol{m}$, $\boldsymbol{jl} = \boldsymbol{n}$, $\boldsymbol{kl} = \boldsymbol{o}$, for reasons of consistency with the Cayley-Dickson form given in § 2.4 giving the multiplication table stated in Table 2.1[3]. This is how octonion multiplication is implemented in [40]. The choice of multiplication table matters only when using quaternions or octonions in component form – most of the formulae in this document are independent of the choice of multiplication table.

The product of two octonions p_1 and p_2, in terms of Cartesian coefficients, using the same con-

[1] We are here treating the quaternion/octonion values as equivalent to *position vectors* in 4- and 8-dimensional Euclidean space, respectively. For a discussion on quaternions as 4-vectors, see [53].

[2] The quaternion multiplication table is not unique (we could replace $\boldsymbol{ijk}$ in (2.1) with $\boldsymbol{kji}$, for example, and thus obtain an opposite-handed basis).

[3] This octonion multiplication table is in agreement with that given in [4, p. 266], and with the historical octonion multiplication tables of Graves (1843) and Cayley [6, Article 21, p. 127]

Table 2.1: Quaternion/octonion multiplication table.

	1	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$	$\boldsymbol{l}$	$\boldsymbol{m}$	$\boldsymbol{n}$	$\boldsymbol{o}$
1	1	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$	$\boldsymbol{l}$	$\boldsymbol{m}$	$\boldsymbol{n}$	$\boldsymbol{o}$
$\boldsymbol{i}$	$\boldsymbol{i}$	-1	$\boldsymbol{k}$	-$\boldsymbol{j}$	$\boldsymbol{m}$	-$\boldsymbol{l}$	-$\boldsymbol{o}$	$\boldsymbol{n}$
$\boldsymbol{j}$	$\boldsymbol{j}$	-$\boldsymbol{k}$	-1	$\boldsymbol{i}$	$\boldsymbol{n}$	$\boldsymbol{o}$	-$\boldsymbol{l}$	-$\boldsymbol{m}$
$\boldsymbol{k}$	$\boldsymbol{k}$	$\boldsymbol{j}$	-$\boldsymbol{i}$	-1	$\boldsymbol{o}$	-$\boldsymbol{n}$	$\boldsymbol{m}$	-$\boldsymbol{l}$
$\boldsymbol{l}$	$\boldsymbol{l}$	-$\boldsymbol{m}$	-$\boldsymbol{n}$	-$\boldsymbol{o}$	-1	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$
$\boldsymbol{m}$	$\boldsymbol{m}$	$\boldsymbol{l}$	-$\boldsymbol{o}$	$\boldsymbol{n}$	-$\boldsymbol{i}$	-1	-$\boldsymbol{k}$	$\boldsymbol{j}$
$\boldsymbol{n}$	$\boldsymbol{n}$	$\boldsymbol{o}$	$\boldsymbol{l}$	-$\boldsymbol{m}$	-$\boldsymbol{j}$	$\boldsymbol{k}$	-1	-$\boldsymbol{i}$
$\boldsymbol{o}$	$\boldsymbol{o}$	-$\boldsymbol{n}$	$\boldsymbol{m}$	$\boldsymbol{l}$	-$\boldsymbol{k}$	-$\boldsymbol{j}$	$\boldsymbol{i}$	-1

ventions for colour, is:

$$p_1p_2 = \begin{aligned} & w_1w_2 - (x_1x_2 + y_1y_2 + z_1z_2 + a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2) \\ & +(w_1x_2 + x_1w_2 + y_1z_2 - z_1y_2 + a_1b_2 - b_1a_2 - c_1d_2 + d_1c_2)\,\boldsymbol{i} \\ & +(w_1y_2 - x_1z_2 + y_1w_2 + z_1x_2 + a_1c_2 + b_1d_2 - c_1a_2 - d_1b_2)\,\boldsymbol{j} \\ & +(w_1z_2 + x_1y_2 - y_1x_2 + z_1w_2 + a_1d_2 - b_1c_2 + c_1b_2 - d_1a_2)\,\boldsymbol{k} \\ & +(w_1a_2 - x_1b_2 - y_1c_2 - z_1d_2 + a_1w_2 + b_1x_2 + c_1y_2 + d_1z_2)\,\boldsymbol{l} \\ & +(w_1b_2 + x_1a_2 - y_1d_2 + z_1c_2 - a_1x_2 + b_1w_2 - c_1z_2 + d_1y_2)\,\boldsymbol{m} \\ & +(w_1c_2 + x_1d_2 + y_1a_2 - z_1b_2 - a_1y_2 + b_1z_2 + c_1w_2 - d_1x_2)\,\boldsymbol{n} \\ & +(w_1d_2 - x_1c_2 + y_1b_2 + z_1a_2 - a_1z_2 - b_1y_2 + c_1x_2 + d_1w_2)\boldsymbol{o} \end{aligned} \tag{2.3}$$

2.1.1 Product in terms of scalar/vector parts

The product of two quaternions or octonions can also be expressed as follows, using scalar and vector parts, and the 'dot' or inner product, and the 'cross' or vector product as given in § 2.8:

$$\begin{aligned} p\,q = & \,\mathrm{S}(p)\,\mathrm{S}(q) - \mathbf{V}(p)\cdot\mathbf{V}(q) + \\ & \mathrm{S}(p)\,\mathbf{V}(q) + \mathrm{S}(q)\,\mathbf{V}(p) + \mathbf{V}(p)\times\mathbf{V}(q) \end{aligned} \tag{2.4}$$

where the first line on the RHS is the scalar part of the product and the second line is the vector part. Notice that if both quaternions/octonions are pure, the RHS reduces to: $-\mathbf{V}(p)\cdot\mathbf{V}(q) + \mathbf{V}(p)\times\mathbf{V}(q)$ which we consider further in § 2.8. If one of the quaternions/octonions is pure, the result reduces to the following (writing the pure quaternion as $\boldsymbol{\rho}$ and choosing this to be the left operand for the sake of demonstration):

$$\boldsymbol{\rho}\,q = -\boldsymbol{\rho}\cdot\mathbf{V}(q) + \mathrm{S}(q)\,\boldsymbol{\rho} + \boldsymbol{\rho}\times\mathbf{V}(q) \tag{2.5}$$

Now assume that $\boldsymbol{\rho} \perp \mathbf{V}(q)$ (a useful special case), then $\boldsymbol{\rho}\cdot\mathbf{V}(q) = 0$, and we have:

$$\boldsymbol{\rho}\,q = \mathrm{S}(q)\,\boldsymbol{\rho} + \boldsymbol{\rho}\times\mathbf{V}(q) \tag{2.6}$$

Notice now that reversing the order of the operands results in negation of the vector part of the result, because reversing the order of operands in a cross product of orthogonal vectors reverses the direction of the result (negates the result in this case). Hence:

$$\boldsymbol{\rho}\,q = \overline{q}\boldsymbol{\rho}, \qquad \text{if} \quad \boldsymbol{\rho}\perp\mathbf{V}(q) \tag{2.7}$$

and similarly if the ordering of the operands on the LHS is reversed. A proof was given for the quaternion case in [32, Appendix C]. It holds for octonions also, as is evident from the above.

2.1.2 Product in terms of orthogonal additive decomposition

A third way to express the product of two quaternions/octonions is by a decomposition into a sum of two orthogonal terms, as follows [25, § 3]:

$$pq = \frac{pq + qp}{2} + \frac{pq - qp}{2} \tag{2.8}$$

The second term here can be recognised as half of the *commutator* of p and q (see § 2.10). The two terms are orthogonal, that is (omitting the common factors of $\frac{1}{2}$):

$$\langle pq + qp, pq - qp\rangle = 0 \tag{2.9}$$

where $\langle x, y\rangle$ is defined in § 2.7.

The half commutator can be recognised as the cross product of the *vector parts* of p and q, as given in (2.36), and as the last term in (2.4). This applies both to the well-known cross product in three dimensions (quaternion case, § 2.8.1), and the binary cross product in seven dimensions (octonion case, § 4.8). In both cases (2.36) holds.

2.2 Scalar and vector parts

The scalar and vector parts (of quaternions and octonions) can be expressed or represented as follows using only arithmetic operations and the conjugate:

$$\mathrm{S}(q) = \tfrac{1}{2}\,(q + \overline{q}) \tag{2.10}$$

$$\mathbf{V}(q) = \tfrac{1}{2}\,(q - \overline{q}) \tag{2.11}$$

These two formulae are perhaps the simplest examples of a more general form $\frac{1}{2}(q \pm f(q))$ where f can be a conjugate as here, an involution (see § 2.11), or more generallly what is known as a *sandwich operator* $x[\ ]y$, where x and y are quaternion constants, and the square brackets indicate a space for the operand[4].

2.3 Coefficients of a quaternion

The following formulae based on involutions (see § 2.11), due to Tony Sudbery, express the four Cartesian coefficients of a quaternion. They would be of value in symbolic algebra, because they express the components using only arithmetic operations. The four coefficients of a quaternion $q = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ are given by [48, Equation 3.1]:

$$w = \tfrac{1}{4}\ \ (q - \boldsymbol{i}q\boldsymbol{i} - \boldsymbol{j}q\boldsymbol{j} - \boldsymbol{k}q\boldsymbol{k}) \tag{2.12}$$

$$x = \tfrac{1}{4\boldsymbol{i}}\,(q - \boldsymbol{i}q\boldsymbol{i} + \boldsymbol{j}q\boldsymbol{j} + \boldsymbol{k}q\boldsymbol{k}) \tag{2.13}$$

$$y = \tfrac{1}{4\boldsymbol{j}}\,(q + \boldsymbol{i}q\boldsymbol{i} - \boldsymbol{j}q\boldsymbol{j} + \boldsymbol{k}q\boldsymbol{k}) \tag{2.14}$$

$$z = \tfrac{1}{4\boldsymbol{k}}\,(q + \boldsymbol{i}q\boldsymbol{i} + \boldsymbol{j}q\boldsymbol{j} - \boldsymbol{k}q\boldsymbol{k}) \tag{2.15}$$

These formulae do not generalise to the octonions, as can easily be confirmed, and the reason is somewhat banal – each monomial term such as $\boldsymbol{i}q\boldsymbol{i}$ negates the scalar part and *one* of the components of the vector part, in this case the coefficient of $\boldsymbol{i}$, leaving the other two unchanged. Since the other two monomial terms do the same, summing the three monomials with the original q results in one component with four terms of the same sign, and three components with two terms of each sign, which cancel out. Applying this to octonions results in one component with eight terms of the same sign, and seven components with two terms of each sign and *six* terms of the same sign, which do not

[4] Sandwich operators can only be needed in a non-commutative algebra, because x and y may not commute with the operand. In the octonion case, parentheses may also be needed to indicate the order of evaluation.

cancel out. Thus the reason the formulae do not generalise has nothing to do with the rather unusual properties of octonions, but everything to do with the difference between 4 and 8 components.

There are four rather more compact formulae for the components of a quaternion based on involutions (see § 2.11). These originated in [12, § 8]. Unlike Sudbery's formulae, they generalise to octonions (in an obvious way).

$$w = \frac{1}{2} (q + \bar{q}) \tag{2.16}$$
$$x = \frac{1}{2\boldsymbol{i}} (q + \boldsymbol{i}\bar{q}\boldsymbol{i}) \tag{2.17}$$
$$y = \frac{1}{2\boldsymbol{j}} (q + \boldsymbol{j}\bar{q}\boldsymbol{j}) \tag{2.18}$$
$$z = \frac{1}{2\boldsymbol{k}} (q + \boldsymbol{k}\bar{q}\boldsymbol{k}) \tag{2.19}$$

In the quaternion case these formulae can be easily expanded to give Sudbery's formulae, by substituting the expression given in (2.50) (a monomial in q) for the conjugates. If you do this substitution in the octonion case, it may be possible to obtain a simplified formula, but not obviously something like Sudbery's formulae. Non-associativity makes the formulae difficult to simplify.

2.4 Cayley-Dickson form or construction

A quaternion in Cartesian form may be factored as follows [23]:

$$q = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = (w + x\boldsymbol{i}) + (y + z\boldsymbol{i})\,\boldsymbol{j} \tag{2.20}$$

Notice that $\boldsymbol{j}$ must be written on the right of the 'complex' number $y + z\boldsymbol{i}$, in order for the product $\boldsymbol{ij}$ to yield $\boldsymbol{k}$ and not $-\boldsymbol{k}$.

(2.20) is usually expressed in reverse: start with two 'complex numbers' and make them the coefficients of a 'new type of complex number', using $\boldsymbol{i}$ for the square root of -1 in the first type of complex number, and $\boldsymbol{j}$ for the square root of -1 in the second type of complex number.

The 'complex numbers' above are actually merely isomorphic to complex numbers. In order for the construction to work, $(w + x\boldsymbol{i})$ must be a quaternion, so that $\boldsymbol{ij}$ yields $\boldsymbol{k}$, that is we cannot use the complex root of -1 denoted by I in this document, in place of the quaternion $\boldsymbol{i}$.

The Cayley-Dickson form can be generalized to an arbitrary orthonormal basis [11, § 4], so that the same quaternion q can be written or represented as:

$$q = (w' + x'\boldsymbol{\mu}_1) + (y' + z'\boldsymbol{\mu}_1)\boldsymbol{\mu}_2 \tag{2.21}$$

where $\boldsymbol{\mu}_1 \perp \boldsymbol{\mu}_2$, with coefficients in a new basis $(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\mu}_1\boldsymbol{\mu}_2)$. The coefficients in the new basis are easily obtained using the scalar product in (2.34), as follows:

$$w' = w \tag{2.22}$$
$$x' = \langle \mathbf{V}(q)\,, \boldsymbol{\mu}_1 \rangle \tag{2.23}$$
$$y' = \langle \mathbf{V}(q)\,, \boldsymbol{\mu}_2 \rangle \tag{2.24}$$
$$z' = \langle \mathbf{V}(q)\,, \boldsymbol{\mu}_1\boldsymbol{\mu}_2 \rangle \tag{2.25}$$

The Cayley-Dickson construction can be used to decompose octonions into a 'complex number' with quaternions[5] as coefficients:

$$p = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} + a\boldsymbol{l} + b\boldsymbol{m} + c\boldsymbol{n} + d\boldsymbol{o} = (w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) + (a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k})\boldsymbol{l} \tag{2.26}$$

For consistency with (2.20) we place $\boldsymbol{l}$ on the right of the second quaternion, but this is an arbitrary choice, leading to one possible multiplication table of many. We expand on this octonion Cayley-Dickson form in § 4.9, because it has applications in the representation of octonions using a pair of quaternions (a useful short-cut in a computer implementation of octonion arithmetic or algebra).

[5] See § 4.9 about this.

The idea of the generalized orthonormal basis extends to octonions, assuming an orthonormal basis can be constructed. The method for obtaining the coefficients in the new basis is the same, but of course the basis has seven mutually orthogonal elements. See § 4.8 for octonion cross products that may be used to construct a basis.

2.5 Roots of -1

Any pure quaternion with real coefficients and unit norm is a square root of -1 [21, pp.203–4, eqns. 340–2]:

$$\boldsymbol{\mu}^2 = -1, \quad \mathrm{S}(\boldsymbol{\mu}) = 0 \quad \text{and} \quad \|\boldsymbol{\mu}\| = 1. \tag{2.27}$$

The case of quaternions with complex coefficients (also known as *biquaternions*) was analysed fully in [37, Theorem 2.1] and the above statement holds in this case also provided that the term *semi-norm* replaces *norm* (the semi-norm is in general complex, and may vanish in the case of divisors of zero). A biquaternion root of -1 is a pure biquaternion, and takes the form:

$$\boldsymbol{\xi} = \pm\left(a\boldsymbol{\mu} + b\boldsymbol{\nu}I\right) \tag{2.28}$$

where $a, b \in \mathbb{R}$, $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are pure unit quaternions with real coefficients (and are therefore themselves roots of -1), with $\boldsymbol{\mu} \perp \boldsymbol{\nu}$, and $a^2 - b^2 = 1$. The constraint on a and b means that they can be represented parametrically by $t \in \mathbb{R}$ such that $a = \cosh^2 t$ and $b = \sinh^2 t$, since $\cosh^2 t - \sinh^2 t = 1$.

Both cases (real and complex coefficients) apply also to octonions.

2.6 Polar form of quaternions and octonions

Quaternions and octonions, like complex numbers, may be written in polar form, based on square roots of -1. The polar form of any quaternion or octonion, with real or complex coefficients (with the exception of divisors of zero), is:

$$q = |q| \exp\left(\boldsymbol{\mu}\theta\right) = |q| \left(\cos\theta + \boldsymbol{\mu}\sin\theta\right) \tag{2.29}$$

where $\boldsymbol{\mu}$ is a unit pure quaternion (and hence $\boldsymbol{\mu}^2 = -1$) usually referred to as the *axis* of the quaternion/octonion, and θ is the angle or argument. This is a quaternion/octonion generalisation of Euler's formula for complex numbers. The axis is analogous to the imaginary axis of the complex plane. Indeed a plane defined by q and $\boldsymbol{\mu}$ is isomorphic to the complex plane and hence it is possible to define a complex number $|q| \exp\left(I\theta\right)$ isomorphic to q with the same modulus and argument (not in the case where q has complex coefficients, obviously). The isomorphic complex number can be used to define/compute functions like the cosine, tangent, square root, *etc.* of the quaternion/octonion. See also § 2.15 for another method of computing such functions.

The axis and angle are given by:

$$\boldsymbol{\mu} = \frac{\mathbf{V}(q)}{|\mathbf{V}(q)|}, \qquad \tan\theta = \frac{|\mathbf{V}(q)|}{\mathrm{S}(q)} \tag{2.30}$$

Note that the angle lies in a half-plane because the modulus of the vector part is conventionally positive. The formula for the axis works in the complex case, but the formula for the angle does not. We give here a formula used in [40, Function: `angle`], based on [29, § 4.6]:

$$\theta = -I \ln\left(\frac{x + Iy}{\sqrt{x^2 + y^2}}\right) \tag{2.31}$$

where $x = \mathrm{S}(q)$ and $y = \|\mathbf{V}(q)\|$; $x, y \in \mathbb{C}$.

2.7 Inner, or scalar products

The scalar, or inner, product of two quaternions is given by [34, Proposition 10.8, p 177]:

$$\langle p, q\rangle = \langle q, p\rangle = \mathrm{S}(\overline{p}q) = \mathrm{S}(\overline{q}p) = \tfrac{1}{2}\left(\overline{p}q + \overline{q}p\right) \tag{2.32}$$

It is well-known that scalar products work in any number of dimensions. The result is a real value equal to the product of the moduli of the two quaternions, multiplied by the cosine of the angle between them. Therefore these formulae are also valid for octonions. In both cases, a zero result indicates that the two values are orthogonal. If p and q have complex coefficients, then the formula will yield a complex value. Expanding and simplifying (2.32) we obtain a formula for direct computation of the scalar product. Writing each quaternion in terms of Cartesian coefficients thus: $p = p_0 + p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k}$, and similarly for q, we obtain:

$$\langle p, q\rangle = \sum_{x=0}^{3} p_x q_x \tag{2.33}$$

This is the preferred formula for numerical or symbolic computation, because it computes only the products which are necessary. In the case of complexified quaternions, the same formula applies, but interpretation of the result is not so straightforward, particularly interpretation of *orthogonality*. For details see [39, § 3.2].

For pure quaternions or octonions $\boldsymbol{f}$ and $\boldsymbol{g}$, the formula reduces to:

$$\langle \boldsymbol{f}, \boldsymbol{g}\rangle = -\tfrac{1}{2}\left(\boldsymbol{f}\boldsymbol{g} + \boldsymbol{g}\boldsymbol{f}\right) \tag{2.34}$$

because conjugation of a pure quaternion/octonion reduces to negation.

2.8 Cross products

We consider here quaternion formulae for cross products in three and four dimensions. Cross products in seven and eight dimensions are considered in § 4.8.

2.8.1 Three dimensions

The classical cross or vector product in 3-dimensions is a part of the full quaternion product in (2.4), but it is clearer if we limit ourselves to pure quaternions $\boldsymbol{f}$ and $\boldsymbol{g}$, where by [34, Proposition 10.15, p 178]:

$$\boldsymbol{f}\boldsymbol{g} = -\boldsymbol{f}\cdot\boldsymbol{g} + \boldsymbol{f}\times\boldsymbol{g} \tag{2.35}$$

from which:

$$\boldsymbol{f}\times\boldsymbol{g} = \tfrac{1}{2}\left(\boldsymbol{f}\boldsymbol{g} - \boldsymbol{g}\boldsymbol{f}\right) = -\boldsymbol{g}\times\boldsymbol{f} \tag{2.36}$$

because the cross product reverses if the order is swapped. It follows that for two orthogonal pure quaternions ($\boldsymbol{f}\perp\boldsymbol{g}$ or equivalently $\boldsymbol{f}\cdot\boldsymbol{g} = 0$):

$$\boldsymbol{f}\boldsymbol{g} = -\boldsymbol{g}\boldsymbol{f} \tag{2.37}$$

The cross product can be computed from a determinant [3, **Vector Product**], as follows:

$$\boldsymbol{f}\times\boldsymbol{g} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{vmatrix} \tag{2.38}$$

where $\boldsymbol{f} = f_x\boldsymbol{i} + f_y\boldsymbol{j} + f_z\boldsymbol{k}$ and similarly for $\boldsymbol{g}$.

The cross product of three pure quaternions obeys the following [34, Proposition 10.18, p 178]:

$$\boldsymbol{f} \times (\boldsymbol{g} \times \boldsymbol{h}) = (\boldsymbol{f} \cdot \boldsymbol{h})\, \boldsymbol{g} - (\boldsymbol{f} \cdot \boldsymbol{g})\, \boldsymbol{h} \tag{2.39}$$

and the Jacobi identity follows:

$$\boldsymbol{f} \times (\boldsymbol{g} \times \boldsymbol{h}) + \boldsymbol{g} \times (\boldsymbol{h} \times \boldsymbol{f}) + \boldsymbol{h} \times (\boldsymbol{f} \times \boldsymbol{g}) = 0 \tag{2.40}$$

For any three quaternions [34, Proposition 10.19, p 179]: $\mathrm{S}(pqr) = \mathrm{S}(prq) = \mathrm{S}(rpq)$, and for any three pure quaternions:

$$\boldsymbol{f} \cdot (\boldsymbol{g} \times \boldsymbol{h}) = -\,\mathrm{S}(\boldsymbol{f}\boldsymbol{g}\boldsymbol{h}) \tag{2.41}$$

This is known as the *scalar triple product*. It can also be computed from a determinant [3, **Triple Product**]:

$$\boldsymbol{f} \cdot (\boldsymbol{g} \times \boldsymbol{h}) = \begin{vmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{vmatrix} \tag{2.42}$$

where again $\boldsymbol{f} = f_x\boldsymbol{i} + f_y\boldsymbol{j} + f_z\boldsymbol{k}$ and similarly for $\boldsymbol{g}$ and $\boldsymbol{h}$. The result is the signed volume of the parallelipiped whose edges are defined by the three pure quaternions.

2.8.2 Four dimensions

It is known that a binary cross product exists only in 3 and 7 dimensions. However, a ternary cross product exists in any number of dimensions, including 4. Thus a ternary cross product of three quaternions exists. Ronald Shaw [46, § 4] provides extensive discussion of this and more, but the clearest formula is in [4, p 270] showing that the determinant formula in (2.38) generalises to four dimensions. In quaternion form the formula, for three full quaternions, is:

$$\times(p, q, r) = \begin{vmatrix} 1 & \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ p_w & p_x & p_y & p_z \\ q_w & q_x & q_y & q_z \\ r_w & r_x & r_y & r_z \end{vmatrix} \tag{2.43}$$

where $p = p_w + p_x\boldsymbol{i} + p_y\boldsymbol{j} + p_z\boldsymbol{k}$, and similarly for q and r. The result of the ternary cross product is invariant to cyclic permutation of the operands, but changes sign if any two are interchanged. Geometrically, the result is orthogonal to the operands. Taking three arbitrary quaternions, assuming no particular geometry, the result will be orthogonal to all three.

2.9 Conjugate rules

The product of two quaternions or octonions obeys the following conjugate rule:

$$\overline{pq} = \overline{q}\,\overline{p} \tag{2.44}$$

from which $pq = \overline{\overline{q}\,\overline{p}}$ *etc.* and this generalizes to more than two terms *in the quaternion case* as follows [24, § 31] or [20, § 20, p 238]:

$$\overline{pqr\ldots} = \ldots\,\overline{r}\,\overline{q}\,\overline{p} \tag{2.45}$$

This last formula cannot work for octonions, except in the special case of a product of two octonions. For a discussion of the rules around conjugates in the octonion case, see § 4.5.

We may also note as a consequence of (2.44) that the inverse of a product is the product of the inverses taken in reverse order (this applies to both quaternions and octonions, and is due to non-commutativity)[6]:

$$(pq)^{-1} = q^{-1}p^{-1} \tag{2.46}$$

[6] Note carefully the following pitfall for the unwary where writing the expression in a different notation can easily lead to a mistake:

$$\frac{1}{pq} \neq \frac{1}{p}\frac{1}{q}.$$

2.10 Commutator

The commutator of two quaternions is [3]:

$$[p, q] = pq - qp. \tag{2.47}$$

This quantity vanishes for any two quaternions p and q that commute because in this case $pq = qp$.

Tevian Dray in his paper [8], [9] says that: the commutator is pure, antisymmetric [that is, $[p, q] = -[q, p]$], and changes sign if one of the arguments is conjugated.

See also § 4.3 for the related concept of the associator, which is not meaningful for quaternions because multiplication is associative.

2.11 Involutions

An involution is in general a self-inverse function or mapping, but in [12] a more specific and useful definition was set out for the case of quaternion involutions (and anti-involutions). The paper did not consider octonions, but in fact the mappings below generalise to octonions, as can easily be verified. An involution $x \mapsto f(x)$ is its own inverse: $f(f(x)) = x$, and is linear: $\lambda f(x_1 + x_2) = \lambda f(x_1) + \lambda f(x_2)$, $\lambda \in \mathbb{R}$. An involution of a product is the product of the involutions of the factors in reverse order: $f(x_1 x_2) = f(x_2) f(x_1)$. In the case of an anti-involution, the involutions of the factors must be in the same order: $f(x_1 x_2) = f(x_1) f(x_2)$.

For any unit pure quaternion $\boldsymbol{\nu}$, the transformation:

$$q \mapsto -\boldsymbol{\nu}\overline{q}\boldsymbol{\nu} \tag{2.48}$$

is an involution that reflects the vector part of q across the *plane* normal to $\boldsymbol{\nu}$. (In the case of octonions, for *plane* read *hyperplane*.) Similarly, the transformation:

$$q \mapsto -\boldsymbol{\nu}q\boldsymbol{\nu} \tag{2.49}$$

(without the conjugate) is an anti-involution that reflects the vector part of q across the *line* defined by the direction of $\boldsymbol{\nu}$. Both transformations leave the scalar part of q unchanged, because the scalar part commutes with $\boldsymbol{\nu}$, and $\boldsymbol{\nu}^2 = -1$, hence applied to a real or complex value, the operation is an identity. The results of the two transformations applied to the same q are quaternion/octonion conjugates of each other.

Involutions provide a way to express the quaternion and octonion conjugates in terms of arithmetic operations alone. Bülow [5, Definition 2.2] was the first to state a formula, as far as we know, which was:

$$\overline{q} = -\tfrac{1}{2}\left(q + \boldsymbol{i}q\boldsymbol{i} + \boldsymbol{j}q\boldsymbol{j} + \boldsymbol{k}q\boldsymbol{k}\right) \tag{2.50}$$

This will also work for octonions with the appropriate amendments. But it also can be generalized using any orthonormal basis in place of $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$, as noted in [12, Theorem 11]. The importance of Bülow's formula, and its generalization, as with Sudbery's formulae for the coefficients of a quaternion in § 2.3, is that they may be useful in symbolic algebra to represent the conjugate operation using only additions and multiplications.

2.12 Quadratic form

For any quaternion q [34, Proposition 10.9, p 178] or [21, Equation 380]:

$$q^2 - (\overline{q} + q)\, q + \overline{q}q = q^2 - 2\,\mathrm{S}(q)\, q + \|q\| = 0. \tag{2.51}$$

This equation is also valid for any octonion, and for quaternions and octonions with complex coefficients.

2.13 Hall identity

The following identity is valid for both quaternions and octonions [16, p 261]:

$$(pq - qp)^2\, r = r(pq - qp)^2 \tag{2.52}$$

including those with complex coefficients. The term in parentheses may be recognised as the commutator of p and q (§ 2.10), hence an alternative expression for the identity is:

$$[p, q]^2\, r = r\, [p, q]^2 \tag{2.53}$$

where $[p, q]$ denotes the commutator of p and q as defined in (2.47). The import of this identity is that the square of any commutator commutes with an arbitrary quaternion or octonion, the reason being that the square of any commutator is a scalar (that is an element of $\mathbb{R}$ or $\mathbb{C}$).

2.14 Polar decomposition

This section is about a factorisation of *complexified* quaternions into a product of two exponentials. It is also valid for complexified octonions. In the case of quaternions with real coefficients, it reduces to the polar form given in § 2.6. The factorisation given here is isomorphic to the polar decomposition of linear algebra [13, § 4.2.10] or [27, § 13.3-4], but of course this isomorphism cannot hold for octonions, because of the lack of a matrix representation of octonion multiplication.

An arbitrary complexified quaternion q with unit semi-norm has the following factorisation [41, Theorem 1]:

$$q = e^{\boldsymbol{\alpha}\theta_t} e^{I\boldsymbol{\beta}\theta_h} \tag{2.54}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are pure real unit quaternions (and therefore $\boldsymbol{\alpha}^2 = \boldsymbol{\beta}^2 = -1$); $I\boldsymbol{\beta}$ is a pure imaginary unit biquaternion (and therefore $(I\boldsymbol{\beta})^2 = +1$). The two angles θ_t and θ_h are real, the subscripts standing for *trigonometric* and *hyperbolic* respectively. The two factors do not commute, so there exists a variant with the factors reversed, and it happens that the trigonometric factor $e^{\boldsymbol{\alpha}\theta_t}$ is invariant to the ordering, but the hyperbolic factor is not.

Computation of the factorisation is simple. The trigonometric factor is given by:

$$e^{\boldsymbol{\alpha}\theta_t} = |\Re\,(q)| \tag{2.55}$$

that is by normalising the real part of the complexified quaternion. The hyperbolic factor is then obtainable by left division of q by the trigonometric factor (right division in the case where the ordering with the hyperbolic factor on the left is to be found).

The special case of divisors of zero is dealt with in [41, § 4], and the case of non-unit semi-norm in [41, Corollary 1].

2.15 Peirce decomposition

The following decomposition of a quaternion with *real* coefficients was published by Roger Oba [33]. Although his paper did not consider octonions, the decomposition is also valid for octonions with real coefficients. It is a Peirce decomposition using idempotents (complexified quaternions or octonions with the property $q^2 = q$).

An arbitrary quaternion or octonion with real coefficients may be decomposed as:

$$q = \chi d^\star + \chi^\star d \tag{2.56}$$

where $\chi \in \mathbb{C}$ is called the positive eigenvalue, and d is the positive idempotent (a complexified quaternion/octonion with real scalar part, and imaginary vector part).

The decomposition has the interesting property that it can be used to evaluate a wide range of functions of a quaternion or octonion variable (for example: powers, trigonometric functions) using only a *complex* computer implementation of the function (see [40, Function: **peirce**]) as follows: $f(q) = f(\chi)d^\star + f(\chi^\star)d$.

χ and d are constructed as follows:

$$d = \frac{1}{2}\left(1 + \frac{\mathbf{V}(q)}{|\mathbf{V}(q)|}I\right) \quad \text{[33, Equations 2.1 and 3.1]} \tag{2.57}$$

$$\chi = \mathrm{S}(q) + |\mathbf{V}(q)|\, I \quad \text{[33, Equation 4.3]} \tag{2.58}$$

Note that there is a more direct way to compute a function of a quaternionic variable using an isomorphic complex number, which can be constructed from the scalar part and modulus of the vector part of a quaternion/octonion [40]. In effect, Oba's result provides a mathematical justification for such an approach, which otherwise seems to be somewhat arbitrary.

2.16 Functions of a quaternion argument

Square roots, exponential, logarithm (here we could usefully refer to the Peirce decomposition of Roger Oba in § 2.15). Trigonometric and hyperbolic functions.

2.17 Campbell-Baker-Hausdorff formula

This is the formula $e^x e^y = e^z$ in a non-commutative algebra. For a general exposition see the Wikipedia article entitled 'Baker-Campbell-Hausdorff formula'. A solution is known for the quaternion case, to be inserted here.

2.18 Geometric components of a biquaternion

The complexified quaternion or biquaternion algebra is isomorphic to the geometric algebra of Euclidean 3-space, or the Clifford algebra with signature $(3, 0)$. This is discussed in [39]. There are formulae in the paper for extracting geometric components using conjugates (Table 4) which we could include here.

2.19 Divisors of zero

These are quaternions/octonions with the property $\|q\| = 0$, even though $q \neq 0$. They divide into idempotents, where $q^2 = q$, and nilpotents, where $q^2 = 0$. See [38]. Brief details to be inserted here.

3 Rotations and reflections

We gather in this chapter material relating to the geometric operations of rotation and reflection, a classical area in which quaternions have been employed, especially in 3-dimensions, although the formulae have been known for a long time even in the case of 4-dimensions. The octonion cases in 7- or 8-dimensions, are less well-known or understood.

Rotations and reflections can be employed to decompose quaternions and octonions into components parallel to and perpendicular to a direction, into orthogonal planes, *etc.* We present some of these in this chapter, because they are based on rotations or reflections. Generalisations of this are called *projection* and *rejection* in geometric algebra.

3.1 Three dimensions

Classical quaternions have long been employed for the representation and computation of rotations, especially in aerospace and rigid body dynamics. The formulae are well known, and given in multiple sources, but the classic paper on this topic is by Coxeter in 1946 [7]. A rotation in three dimensions is expressed by the following mapping:

$$\boldsymbol{x} \mapsto p\boldsymbol{x}\overline{p} \tag{3.1}$$

where $\boldsymbol{x}$ is a pure quaternion representing a direction in 3-space, and the full quaternion p expresses both the axis of rotation (by the direction of its vector part), and the angle of rotation around the axis (by half the angle of p – see § 2.6). A special case is a reflection in a plane, which is given by [7, Theorem 3.1] and [34, Proposition 10.21, p 180]:

$$\boldsymbol{x} \mapsto \boldsymbol{y}\boldsymbol{x}\boldsymbol{y}^{-1} \tag{3.2}$$

where $\boldsymbol{x}$ is a pure quaternion, and $\boldsymbol{y}$ is a pure quaternion normal to the plane of reflection. The result is a pure quaternion of the same modulus as $\boldsymbol{y}$, reflected in the plane, so that the projections of $\boldsymbol{y}$ and its reflection onto the plane are identical, while the components of $\boldsymbol{y}$ and its reflection along the line normal to the plane are oppositely directed. It follows that we can decompose $\boldsymbol{y}$ into components normal to and parallel to the plane as follows [12, Theorem 12]:

$$\begin{aligned} \boldsymbol{x}_\parallel &= \tfrac{1}{2}\left(\boldsymbol{x} - \boldsymbol{y}\boldsymbol{x}\boldsymbol{y}^{-1}\right) \\ \boldsymbol{x}_\perp &= \tfrac{1}{2}\left(\boldsymbol{x} + \boldsymbol{y}\boldsymbol{x}\boldsymbol{y}^{-1}\right) \end{aligned} \tag{3.3}$$

such that $\boldsymbol{x} = \boldsymbol{x}_\perp + \boldsymbol{x}_\parallel$, and $\boldsymbol{x}_\perp$ is orthogonal to both $\boldsymbol{x}_\parallel$ and $\boldsymbol{y}$, that is: $\langle \boldsymbol{x}_\perp, \boldsymbol{x}_\parallel \rangle = \langle \boldsymbol{x}_\perp, \boldsymbol{y} \rangle = 0$.

These formulae are an example of a more general case discussed earlier in § 2.2. Since $\boldsymbol{y}$ is a pure quaternion, we can simplify these formulae by replacing $\boldsymbol{y}$ with a *unit* pure quaternion, say $\boldsymbol{\mu} = \boldsymbol{y}/|\boldsymbol{y}|$, in which case $\boldsymbol{\mu}^{-1} = -\boldsymbol{\mu}$, $\|\boldsymbol{\mu}\| = 1$, and our formulae become:

$$\begin{aligned} \boldsymbol{x}_\parallel &= \tfrac{1}{2}\left(\boldsymbol{x} + \boldsymbol{\mu}\boldsymbol{x}\boldsymbol{\mu}\right) \\ \boldsymbol{x}_\perp &= \tfrac{1}{2}\left(\boldsymbol{x} - \boldsymbol{\mu}\boldsymbol{x}\boldsymbol{\mu}\right) \end{aligned} \tag{3.4}$$

We can recognise the operation $\boldsymbol{x} \mapsto -\boldsymbol{\mu}\boldsymbol{x}\boldsymbol{\mu}$ as the anti-involution given in (2.49).

3.2 Four dimensions

Note that rotations in four dimensions offer more possibilities than in three. See Wikipedia: Rotations in 4-dimensional Euclidean space. As well as rotations about a point/line/plane, it is possible to rotate about *two* planes (a *double* rotation).

Formulae for rotations in four dimensions are given by Coxeter [7, § 5], but see also the thesis of Johan Mebius (1994) [30, Chapter 3].

3.3 Seven and eight dimensions

Rotations in 7 and 8 dimensions using octonions are covered in [50]. These are rotations of the special orthogonal group SO(7), and don't include double or triple(?) rotations, so some further research is needed to locate material on these.

See also [55] for a discussion about reflections in 8 dimensions.

4 Octonion-specific formulae

In this section we present formulae that apply only to octonions (mainly because they relate to non-associativity, or to 7- and 8-dimensions).

4.1 Bi-associativity

We stated in the introduction to this compendium that octonions are not associative, but there are special cases where a product of three or more octonions can be evaluated in arbitrary order (that is, parentheses are superfluous) yielding the same result for any order of evaluation. This is an important algebraic tool, and it takes the form of a rule rather than a formula, nevertheless it is sufficiently important to present it at the start of this section on octonion formulae.

Definition 1 (Bi-associativity)**.** *The order of evaluation does not affect the result (and therefore parentheses are not needed) in any multiple product whose terms are drawn from:* p, q, $\overline{p}$, $\overline{q}$, p^{-1}, q^{-1}, *or* $r \in \mathbb{R}$ *or* $z \in \mathbb{C}$ *where* p *and* q *are arbitrary octonions [36, § 11.9].*

Note carefully that in Definition 1 the only conjugates permitted are octonion conjugates, and specifically *not* complex conjugates[1].

Ward [54, p. 179] states the above result in a slightly more general way paraphrased as: if any two of the octonions have parallel vector parts, the product is associative.

Some authors (for example Schafer [43, pp 27–28]) quote the so-called *alternative laws*:

$$p^2 q = p(pq) \tag{4.1}$$

$$p(qp) = (pq)p \tag{4.2}$$

$$qp^2 = (qp)p \tag{4.3}$$

although **for octonions** they follow directly from the more general case in Definition 1. Another useful result that follows directly from Definition 1 is:

$$p^{-1}(pq) = q = (qp)\,p^{-1} \tag{4.4}$$

4.2 The Moufang identities

Theorem 1 (Moufang identities)**.** *The following identities are valid for arbitrary octonions* p, q, r *[43, p 28] (or [42, p 18]) and are due to Ruth Moufang (1905–1977) [31].*

$$\begin{aligned} r(q(rp)) &= ((rq)r)p \\ &= (rqr)p \end{aligned} \tag{4.5}$$

$$\begin{aligned} ((pr)q)r &= p(r(qr)) \\ &= p(rqr) \end{aligned} \tag{4.6}$$

$$(rp)(qr) = r(pq)r \tag{4.7}$$

Schafer [42, p 18] proves these identities using associators (see § 4.3). The first identity requires seven steps starting from $(rqr)p - r(q(rp)) = 0$ and ending with $-r(-[q, r, p] + [p, q, r]) = 0$ which is evidently true because the two associators have the same sign and value as a consequnce of cyclic permutation of the operands. The term in parentheses is therefore zero.

[1] Or a combined conjugation of an octonion conjugate and a complex conjugate.

The first two identities are known as the *left* and *right identities* because the terms are in identical sequence on the left and right sides, but the order of evaluation on the left is the reverse of that on the right. The right-hand sides of the first two identities do not need parentheses within the term rqr because of Definition 1 (only two distinct variables appear in the triple product), which is why the RHS is written in two different ways.

Similarly the right-hand side of the last identity does not need outer parentheses, since after first evaluating (pq) there are only two distinct terms remaining, and therefore the result is not dependent on whether the left or right r is multiplied first.

One can use inverses to rewrite the left and right Moufang identities in a more useful form (see Wikipedia — Moufang loop):

$$(pq)r = (pr^{-1})(rqr) \tag{4.8}$$

$$r(qp) = (rqr)(r^{-1}p) \tag{4.9}$$

For a proof of the first of these two alternate formulae see § A.1. An original source for the alternate forms of the identities is not known.

4.3 Associator

The associator of three octonions is defined as [2]:

$$[p, q, r] = p(qr) - (pq)r \tag{4.10}$$

This expresses the difference between two products which differ in the order in which the two multiplications are carried out. Compare with the commutator in § 2.10. The associator is invariant to cyclic permutation of the variables, but it changes sign if the permutation is odd [42, p. 17]. Thus, for example, $[p, q, r] = [r, p, q]$, but $[p, q, r] = -[q, p, r]$. The associator is pure, antisymmetric, and changes sign if any one of the arguments is conjugated [9, p. 6] and [8].

Any four octonions p, q, r, s satisfy the following *associator identity* [43, § 2.4, p. 13], [42, Equation 12, p. 10]:

$$s[p, q, r] + [s, p, q]r = [sp, q, r] - [s, pq, r] + [s, p, qr]. \tag{4.11}$$

The following formula is given by Bremner and Hentzel [4, p265] and holds for any alternative algebra. It expresses the associator in terms of a sum of nested commutators (see § 2.10). It has been verified for octonions, including complexified octonions.

$$6[a, b, c] = [[a, b], c] + [[b, c]a] + [[c, a], b] \tag{4.12}$$

In § 2.1.2, (2.8), we saw how the octonion product could be expressed in terms of an additive decomposition of two orthogonal terms based on the commutator [25, § 3]. Likewise, the octonion triple product can be expressed as the sum of three additive mutually orthogonal terms [25, § 4]. Let p, q and r be three arbitrary full octonions. Then the following holds (notice that the central octonion in the product is conjugated):

$$(p\overline{q})r = \frac{(p\overline{q})r + (r\overline{q})p}{2} + \frac{(p\overline{q})r - r(\overline{q}p)}{2} + \frac{r(\overline{q}p) - (r\overline{q})p}{2} \tag{4.13}$$

There is more to be said about this decomposition, for which see [25, § 4 and § 5], but notice here that the middle term above is the ternary cross product (4.19) which we consider below in § 4.8.2. For the moment, only the basic formula has been quoted here, and the full connection with the associator has not yet been made clear. The original source is open access, so the reader can easily consult it for more details.

4.4 Kleinfeld product of four octonions

The following product of four octonions, based on associators, has the property that it vanishes if any two of the four octonions are identical in value [26]:

$$[wx, y, z] - x[w, y, z] - [x, y, z]w \tag{4.14}$$

4.5 Conjugate rules

The conjugate rules in § 2.9 cannot be applied to a product of more than two octonions because of non-associativity. We present here more of a discussion than an exhaustive analysis of the possibilities. Consider, for example, a nested product, which is the simplest generalisation of the quaternion generalised conjugate rule in (2.45):

$$\overline{((pq)r)\ldots} = \ldots\,(\overline{r}\,(\overline{q}\,\overline{p})) \tag{4.15}$$

which can be extended without limit. However, even for a product of four octonions, other possible orders of evaluation exist:

$$\overline{(sp)(qr)} = (\overline{r}\,\overline{q})(\overline{p}\,\overline{s})$$

A paper by Bremner and Hentzel [4, p266] contains some material which may be pertinent to this issue.

4.6 Scalar triple product

There is a generalisation of the quaternion scalar triple product given in (2.41) [34, Proposition 14.5, p 280], for $p, q, r \in \mathbb{O}$:

$$\overline{p} \cdot qr = \overline{q} \cdot rp = \overline{r} \cdot pq \tag{4.16}$$

This has been verified symbolically to work for arbitrary octonions using the octonion symbolic computation capability in QTFM [40]. Since no restriction to real coefficients was imposed the result shows that this formula is valid for complexified octonions.

Insert some interpretation of this scalar triple product (source unknown as yet).

4.7 Associative 3-form

Tevian Dray cites in his papers [9, Eqn 13] or [8] the following, which like the associator is antisymmetric and changes sign if any one of the arguments is conjugated:

$$\Phi(p, q, r) = \tfrac{1}{2}\,\mathrm{S}(p(\overline{q}r) - r(\overline{q}p)) \tag{4.17}$$

which reduces to the vector triple product (which?) when the three octonions are pure (check):

$$\Phi(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) = \tfrac{1}{2}\,\mathrm{S}([\boldsymbol{p}, \overline{\boldsymbol{q}}]\,\boldsymbol{r}) \tag{4.18}$$

This associative 3-form is the scalar part of both the middle term in the additive decomposition (4.13) and of the ternary cross product in (4.19).

Dray's paper is not the original source of the above. He quotes two sources, one of which is [35]. This should be checked and the above referenced to the original source.

4.8 Cross products

Three and four dimensional quaternion cross products were covered in § 2.8. Similarly to the quaternion case, a binary cross product exists in 7-dimensions, for pure octonions; and a ternary cross product exists in 8-dimensions for full octonions. Unlike the quaternion case, these products cannot be formulated as determinants because the 7-dimensional case requires 7 columns but only 3 rows, and the 8-dimensional case requires 8 columns and three rows. Neither of these is square, which is a requirement of a determinant.

4.8.1 Seven dimensions

A binary cross product exists in seven dimensions. This is the only other case in Euclidean space, apart from three dimensions, where a binary cross product can exist [28, Theorem 1]. The formulae given in (2.35) and (2.36) hold for octonions. Therefore a simple but inefficient way to compute the 7-dimensional cross product is to compute the full product of the two pure octonions and discard the scalar part of the result[2]. It is also possible to compute the seven-dimensional cross product using the commutator, as described for both quaternions and octonions in § 2.1.2.

4.8.2 Eight dimensions

In eight dimensions, no binary cross product can exist, but as in the case of 4-dimensions and full quaternions, a ternary cross product of three full octonions does exist. Multiple authors have addressed this issue including Ronald Shaw [44], [45], [46]. but it suffices here to quote formulae from two others [35], [57].

F. Reese Harvey [35, Definition 6.54] gives the following formula for a *triple* (ternary) cross product:

$$\times(p, q, r) = \tfrac{1}{2}\left[p\,(\overline{q}r) - r\,(\overline{q}p)\right] \tag{4.19}$$

Peter Zvengrowski [57, Theorem 2.1] (published much earlier), gives this formula:

$$\times(p, q, r) = -p\,(\overline{q}r) + p\,(q \cdot r) - q\,(r \cdot p) + r\,(p \cdot q) \tag{4.20}$$

where the centred dot represents the scalar product. The two formulae are equivalent up to sign[3].

Zvengrowski's formula appears to be subtracting from the first term in Harvey's formula those parts that are cancelled out by the second term. It is therefore a better formula to use for computation, although there is possibly an even better formula that avoids calculating the parts that are subtracted.

Note that in 8-dimensions, there are many possible ternary cross products possible (in the sense that permutations of ordering in Harvey's formula will yield valid but different results). This is easily understood from the dimensionality: orthogonal to any three arbitrary octonions in which no pair is co-planar, there is a 5-dimensional space. Therefore there is not a unique orthogonal result.

4.9 The Cayley-Dickson form and product

Here we consider the product of two arbitrary octonions expressed using the Cayley-Dickson form, because of its utility in computer implementations of octonion multiplication.

Equation (2.26) showed how an octonion can be represented in Cayley-Dickson form analogous to a complex number with 'quaternions' as coefficients. In fact the 'quaternions' must be octonions with values in a quaternion sub-space, specifically in this context, with non-zero real or complex numbers only in the scalar part, and as the coefficients of the *octonion* $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ basis elements. Any such 'quaternion' is orthogonal to the octonion basis element $\boldsymbol{l}$, or indeed any pure octonion with zero coefficients of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. The product of any two such 'quaternions' is also a 'quaternion', and therefore the orthogonality condition just stated still holds for the product.

[2] The computation of the unused scalar part represents only $\frac{1}{8}^{\text{th}}$ of the full product computation.

[3] Verified using the Quaternion Toolbox for MATLAB® [40].

4.9.1 Some useful preliminary results

Baez [2, Equations (4)] gives the following three formulae[4], which we will find useful in proving the result in the following sub-section. Given two arbitrary 'quaternions' p and q in the sense described above, and a unit pure octonion $\boldsymbol{\nu}$ (such that $\boldsymbol{\nu}^2 = -1$ and $\boldsymbol{\nu}^{-1} = -\boldsymbol{\nu}$) orthogonal to both p and q, and to pq (that is $\langle p, \boldsymbol{\nu}\rangle = \langle q, \boldsymbol{\nu}\rangle = \langle pq, \boldsymbol{\nu}\rangle = 0$), we have that:

$$p(\boldsymbol{\nu} q) = \boldsymbol{\nu}(\overline{p}q), \tag{4.21}$$

$$(p\boldsymbol{\nu})q = (p\overline{q})\boldsymbol{\nu}, \tag{4.22}$$

$$(\boldsymbol{\nu} p)(q\boldsymbol{\nu}^{-1}) = \overline{p\,q}. \tag{4.23}$$

Proofs of these three identities are given in § A.2.1.

4.9.2 The octonion product expressed in Cayley-Dickson form

The octonion product may be expressed in Cayley-Dickson form as:

$$(a + b\boldsymbol{l})(c + d\boldsymbol{l}) = (ac - \overline{d}b) + (b\overline{c} + da)\boldsymbol{l} \tag{4.24}$$

where a, b, c and d are 'quaternions' in the same sense as p and q in the previous sub-section. A proof of the formula is given in § A.2.2.

Variations on this formula exist in the literature, for example in Ward's book [54, p. 169] and in the article by Baez [2, p. 154], almost certainly due to differences in the way the Cayley-Dickson form is constructed (for example, $\boldsymbol{l}$ could be placed on the left instead of the right of the quaternions, which would change the formula and the octonion multiplication table). The octonion product in [40] is based on the formula above because it implements the product in terms of quaternion operations which were already coded and validated, avoiding an elaborate direct implementation.

[4] The way the formulae are expressed here may be more specific than Baez, as he was writing about the Cayley-Dickson construction in rather more abstract terms than are employed here, in order to discuss the loss of properties such as commutativity, associativity, at each step in a Cayley-Dickson doubling process.

5 Matrix representations of quaternions

In this chapter we consider two related issues. One is the representation of quaternion multiplication by matrices, and the other is the representation of quaternion products by matrix/vector products. The latter permits a quaternion product pq to be represented in two ways, using a matrix representation of p and a column vector to represent q, or *vice versa*. Source material for this chapter includes [56] and [22]. There are some much older papers which can be cited here, including: [10] and [47].

6 Matrix-vector representation of octonions

It is clear from the non-associativity of octonion multiplication that there cannot exist a matrix representation for octonions similar to that for quaternions, in which the multiplication of matrices would be equivalent to multiplication of octonions represented by the matrices.

However, this does not preclude a representation of octonion multiplication using matrices and vectors, and that is what we present in this chapter.

See Tian's paper [52] and [51] but also [15] which seems to use the same representation with the fairly trivial modification that the octonions are integral octonions.

A Derivations and proofs of selected octonion formulae

We include here some derivations and proofs of octonion formulae. Generally speaking in this document, proofs and derivations have been omitted, because they are available in the cited references, and it is not necessary to read a proof in order to use a formula. However, there are a few cases where the formula presented is not at all obvious, and as far as can be checked there is no published source that derives the result explicitly. In these cases, we include a demonstrations or proof here. These proofs also serve to demonstrate the difficulty of manipulating octonion expressions, which is why they are written in a tutorial, rather than concise, style. There is more about this in Appendix B.

A.1 Alternate forms of the first two Moufang identities

Recall the two forms of the first Moufang identity from Chapter 4:

$$r(q(rp)) = (rqr)p \tag{4.5}$$

$$(pq)r = (pr^{-1})(rqr) \tag{4.8}$$

We show below that these are equivalent by algebraic manipulation of (4.5) into (4.8), making use of the third Moufang identity and bi-associativity. The first step is by no means obvious, and was discovered only after considerable effort[1] – multiply both sides of (4.5) on the right by (qr):

$$(r(q(rp)))(qr) = ((rqr)p)(qr) \tag{A.1}$$

The sequence of variables on each side now starts with r, q and ends with q, r, which allows us to use the third Moufang identity multiple times on the LHS (leaving the RHS untouched for the moment). Using the identity has the effect of 'moving the parentheses', without altering the sequence of variables r, q, r, p, q, r. Recalling that the third Moufang identity (4.7) takes the form $(xy)(zx) = x(yz)x$ and identifying $x = r$, $y = q(rp)$, and $z = q$, application of the identify gives:

$$r(q(rp)q)r = ((rqr)p)(qr) \tag{A.2}$$

Notice that in the line above, we have been able to write $q(rp)q$ in place of $(q(rp))q$ by bi-associativity, since there are only two terms in the expression: q and (rp). Now we apply the third Moufang identity on the LHS again to the sub-expression $q(rp)q$, this time using the identity in reverse:

$$r((qr)(pq))r = ((rqr)p)(qr) \tag{A.3}$$

The third application of the identity is in the same sense as the first, and yields:

$$(rqr)((pq)r) = ((rqr)p)(qr) \tag{A.4}$$

[1] The considerable effort may be due to the author's lack of algebraic ability, so perhaps the reader will be able to discover an alternative (the sequence of manipulations given here is unlikely to be unique). Do please contact me if you discover another starting point and sequence of steps, and are willing to contribute it to the paper. or can refer me to a published source.

where we have also used bi-associativity to write $r(qr)$ as rqr. Now we introduce unity on the RHS in the form of $(r^{-1}r) = 1$. (This is a second non-obvious step.) We place unity inside the parentheses to the left of qr, and since $r^{-1}r \in \mathbb{R}$, by bi-associativity, we need no parentheses:

$$(rqr)((pq)r) = ((rqr)p)(r^{-1}rqr) \tag{A.5}$$

We can then apply the third Moufang identity on the entire RHS, identifying $x = rqr$, $y = p$, and $z = r^{-1}$, yielding:

$$(rqr)((pq)r) = (rqr)(pr^{-1})(rqr) \tag{A.6}$$

Finally we multiply both sides on the left by $(rqr)^{-1}$:

$$(rqr)^{-1}\left[(rqr)((pq)r)\right] = (rqr)^{-1}\left[(rqr)(pr^{-1})(rqr)\right] \tag{A.7}$$

and by bi-associativity we are able to remove the square brackets on both sides. The LHS then takes the form $x^{-1}xy = y$ and the RHS takes the form $x^{-1}xzx = zx$, where $x = (rqr)$, $y = (pq)r$ and $z = (pr^{-1})$. Hence we obtain:

$$(pq)r = (pr^{-1})(rqr) \tag{4.8}$$

We assume from symmetry that a similar sequence of steps with opposite 'handedness' would apply to the alternate form of the second Moufang identity (4.6).

A.2 Octonion multiplication in Cayley-Dickson form

A.2.1 The identities in § 4.9.1

Proof of identity (4.21).

$$p(\boldsymbol{\nu} q) = \boldsymbol{\nu}(\bar{p}q) \tag{4.21}$$

By (4.9) we may rewrite the left-hand side as: $p(\boldsymbol{\nu} q) = (p\boldsymbol{\nu} q)(p^{-1}q)$. We may then change the order of the terms in the first parenthesis on the right using (2.7): $(p\bar{p}\boldsymbol{\nu})(p^{-1}q) = \|p\|\,\boldsymbol{\nu}(p^{-1}q)$. Since the norm is real (or complex), it commutes and associates with all the other terms, thus we may move it into the parentheses, giving: $\boldsymbol{\nu}(\|p\|\,p^{-1}q) = \boldsymbol{\nu}(\bar{p}q)$. □

A proof of (4.22) would be similar.

Proof of identity (4.23).

$$-(\boldsymbol{\nu} p)(q\boldsymbol{\nu}) = \overline{p\,q}, \qquad \text{(writing } \boldsymbol{\nu}^{-1} \text{ as } -\boldsymbol{\nu}) \tag{4.23}$$

Apply the third Moufang identity (4.7) to the LHS:

$$-(\boldsymbol{\nu}(pq)\boldsymbol{\nu}) = \overline{p\,q} \tag{A.8}$$

Then, recalling that pq and $\boldsymbol{\nu}$ are orthogonal, by (2.7):

$$-\boldsymbol{\nu}^2\overline{(pq)} = \overline{p\,q} \tag{A.9}$$

and the LHS is equal to the RHS because $-\boldsymbol{\nu}^2 = +1$. □

A.2.2 The octonion product formula in § 4.9.2

Proof of (4.24)*.* Multiply out the left-hand side of the equation as follows (inserting parentheses as required to ensure the correct order of evaluation):

$$(a + b\boldsymbol{l})(c + d\boldsymbol{l}) = ac + (b\boldsymbol{l})(d\boldsymbol{l}) + (b\boldsymbol{l})c + a(d\boldsymbol{l}) \tag{A.10}$$

The second term on the right can be re-ordered by (2.7):

$$= ac + (\boldsymbol{l}\bar{b})(d\boldsymbol{l}) + (b\boldsymbol{l})c + a(d\boldsymbol{l}) \tag{A.11}$$

then by the third Moufang identity (4.7):

$$= ac + \boldsymbol{l}(\bar{b}d)\boldsymbol{l} + (b\boldsymbol{l})c + a(d\boldsymbol{l}) \tag{A.12}$$

then using biassociativity and (2.7) again, we obtain:

$$= ac + \boldsymbol{l}^2(\overline{\bar{b}d}) + (b\boldsymbol{l})c + a(d\boldsymbol{l}) \tag{A.13}$$

and finally using the conjugate rule (2.44) and $\boldsymbol{l}^2 = -1$, we get:

$$= ac - \bar{d}b + (b\boldsymbol{l})c + a(d\boldsymbol{l}) \tag{A.14}$$

The third term on the right can be re-ordered using (4.22) to put $\boldsymbol{l}$ on the right:

$$= ac - \bar{d}b + (b\bar{c})\boldsymbol{l} + a(d\boldsymbol{l}) \tag{A.15}$$

and the fourth term on the right can be re-ordered to put $\boldsymbol{l}$ on the right, by application of (2.7), the conjugate rule (2.44), and (4.21):

$$= ac - \bar{d}b + (b\bar{c})\boldsymbol{l} + a(\boldsymbol{l}\bar{d}) \qquad \text{by (2.7)} \tag{A.16}$$

$$= ac - \bar{d}b + (b\bar{c})\boldsymbol{l} + \boldsymbol{l}(\bar{a}\bar{d}) \qquad \text{by (4.21)} \tag{A.17}$$

Then since the product of a and d is in the same quaternion sub-space as both, it is therefore also orthogonal to $\boldsymbol{l}$. Hence we can employ (2.7) again to obtain:

$$= ac - \bar{d}b + (b\bar{c})\boldsymbol{l} + (\overline{\bar{a}\bar{d}})\boldsymbol{l} \tag{A.18}$$

$$= ac - \bar{d}b + (b\bar{c})\boldsymbol{l} + (da)\boldsymbol{l} \qquad \text{by (2.44)} \tag{A.19}$$

and finally we group the terms into the two Cayley-Dickson quaternion components:

$$= (ac - \bar{d}b) + (b\bar{c} + da)\boldsymbol{l} \tag{4.24}$$

□

B A tutorial on manipulation of octonion expressions

Manipulation of octonion expressions using algebraic steps is significantly more complicated than manipulation of quaternion expressions. The lack of associativity (or put another way, the need to use parentheses extensively), means that you cannot make progress by working with variables one at a time (a few algebraic experiments on paper should convince you of this). So, what are the extra techniques needed on top of those required by non-commutativity? Here is a list, maybe not exhaustive:

- Work with composite terms, rather than individual variables. This is especially the case to move a parenthesised term from one side of an equation to the other.
- Multiply on the left by a term that already appears on the right, or *vice versa*, so that the third Moufang identity in (4.7) can be brought into play, to enable a parenthesised product to be split, or merged.
- Use the conjugate/inverse rule to re-order terms, and thus allow the third Moufang identity to be used.
- Introduce unity into an expression, in the middle, in the form $x^{-1}x$ or xx^{-1}. Notice that unity is a scalar, so we can introduce it anywhere, *and then* place the parentheses as we wish, grouping unity with terms to the left or the right. What you can't do is split unity with a pair of parentheses like this: $\ldots x)(x^{-1}\ldots$. For an example of the use of this technique see (A.5) and preceding text.

Finally, beware of the need to insert parentheses when multiplying by a variable or term that is not already present. Associativity is hard to unlearn: you need to be alert to mistakes caused by forgetting that parentheses are needed. For example, if you have $x(yz)$ and then you multiply on the left by w, you must insert parentheses like this: $w(x(yz))$. This may seem easy enough, but when x, y and z are themselves composite terms or have exponents such as -1 present, it is all too easy to forget.

In the following sections we present some manipulations to illustrate the above.

B.1 Proof of equation (4.5)

The first of the three Moufang identities (4.5) can be proved, provided that a proof of the third Moufang identity in (4.7) exists, because the identity is used twice below. (The proof below may not be the shortest way to prove this identity.) We start with (4.5) as follows:

$$r(q(rp)) = (rqr)p \tag{4.5}$$

The first step is to multiply both the LHS and the RHS on both sides by r^{-1}, which makes it possible to separate the terms on the RHS:

$$(q(rp))\,r^{-1} = r^{-1}\,[(rqr)p]\,r^{-1} \tag{B.1}$$

Now using (4.7) on the RHS, followed by bi-associativity, we obtain:

$$(q(rp))\,r^{-1} = \left[r^{-1}(rqr)\right]\left(pr^{-1}\right) = (qr)(pr^{-1}) \tag{B.2}$$

Multiply on the left by $(qr)^{-1}$ so that the q's are on the same side:

$$(qr)^{-1}\left[(q(rp))\,r^{-1}\right] = pr^{-1} \tag{B.3}$$

then invert the first term on the left using (2.46):

$$\left[r^{-1}q^{-1}\right]\left[(q(rp))r^{-1}\right] = pr^{-1} \tag{B.4}$$

and then apply (4.7) on the left:

$$r^{-1}(q^{-1}(q(rp)))r^{-1} = pr^{-1} \tag{B.5}$$

then by bi-associativity:

$$r^{-1}(rp)r^{-1} = pr^{-1} \tag{B.6}$$

and by bi-associativity again, the LHS equals the RHS.

We can prove the alternative form of the above formula (4.8), using the same three techniques (inverses, the third Moufang identity, and bi-associativity in the final step):

$$(pq)r = (pr^{-1})(rqr) \tag{B.7}$$

$$(pr^{-1})^{-1}((pq)r) = (rqr) \tag{B.8}$$

$$(rp^{-1})((pq)r) = (rqr) \tag{B.9}$$

$$r(p^{-1}(pq))r = (rqr) \tag{B.10}$$

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